The fifth-generation (G5) CT scanners eliminate all mechanical motion by employing electron beams controlled electromagnetically.

The sixth-generation (G6) CT scanners rotate the source-detector pair continuously through $360^\circ$, while the patient is moved at a constant speed along the axis perpendicular to the scan.

The seventh-generation (G7) CT scanners (also called multislice CT scanners) use parallel banks of detectors to collect volumetric CT data simultaneously.

**Projections and the Radon Transform**

A straight line in Cartesian coordinates can be described either by its slope-intercept form

$$y = ax + b,$$

or, as in Figure 5.36, by its normal representation

$$x \cos \theta + y \sin \theta = \rho.$$  \hspace{1cm} (5.11-1)

![Figure 5.36 Normal representation of a straight line.](image)
The projection of a parallel-ray beam may be modeled by a set of such lines, as shown in Figure 5.37.

![Figure 5.37 Geometry of a parallel-ray beam.](image)

An arbitrary point in the projection signal is given by the raysum along the line

\[ x \cos \theta_k + y \sin \theta_k = \rho_j. \]

In the case of continuous, the raysum is a line integral, given by

\[ g(\rho_j, \theta_k) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta_k + y \sin \theta_k - \rho_j) dx dy \quad (5.11-2) \]

Recall the properties of the impulse, \( \delta \), the right side of (5.11-2) is zero unless the argument of \( \delta \) is zero. It indicates that the integral is computed only along the line \( x \cos \theta_k + y \sin \theta_k = \rho_j \).
If we consider all values of $\rho$ and $\theta$, (5.11-2) generalizes

$$g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho) dx dy.$$  \hspace{1cm} (5.11-3)

The equation (5.11-3) gives the projection of $f(x, y)$ along an arbitrary line in the $xy$-plane, is called the Radon transform.

The Radon transform is the cornerstone of reconstruction from projections, with CT being its principle application in the field of image processing.

In the discrete case, (5.11-3) becomes

$$g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho),$$ \hspace{1cm} (5.11-4)

where $x$, $y$, $\rho$, and $\theta$ are now discrete variables.

If we fix $\theta$ and allow $\rho$ to vary, (5.11-4) simply sums the pixels of $f(x, y)$ along the line defined by specified values of these two parameters.

Incrementing through all values of $\rho$ required to span the image (with $\theta$ fixed) yields one projection. Changing $\theta$ and repeating the same procedure will yield another projection.
Example 5.17: Using the Radon transform to obtain the projection of a circular region.

We want to obtain the Radon transform for the projection of the circular object

\[ f(x, y) = \begin{cases} 
A & x^2 + y^2 \leq r^2 \\
0 & \text{otherwise}
\end{cases}, \]

where \( A \) is a constant and \( r \) is the radius of the object. The circular object is shown in Figure 5.38 (a).

Since the object is circularly symmetric, its projections are the same for all angles, so all we need is to obtain the projection for \( \theta = 0^\circ \). From (5.11-3), we get

\[
g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\delta(x - \rho) \, dx \, dy
\]

\[
= \int_{-\infty}^{\infty} f(\rho, y) \, dy
\]

This is a line integral along the line \( L(\rho, 0) \).
Note that \( g(\rho, \theta) = 0 \) when \( |\rho| > r \). When \( |\rho| \leq r \), the integral is evaluated from \( y = -\sqrt{r^2 - \rho^2} \) to \( y = \sqrt{r^2 - \rho^2} \).

Therefore,

\[
g(\rho, \theta) = \int_{-\sqrt{r^2 - \rho^2}}^{\sqrt{r^2 - \rho^2}} f(\rho, y) dy = \int_{-\sqrt{r^2 - \rho^2}}^{\sqrt{r^2 - \rho^2}} A dy
\]

It yields

\[
g(\rho, \theta) = g(\rho) = \begin{cases} 2A\sqrt{r^2 - \rho^2} & |\rho| \leq r \\ 0 & \text{otherwise} \end{cases}
\]

Figure 5.38 (b) shows the result.

\( g(\rho, \theta) = g(\rho) \) indicates that \( g \) is independent of \( \theta \) because the object is symmetric about the origin.
When the Radon transform, \( g(\rho, \theta) \), is displayed as an image with \( \rho \) and \( \theta \) as rectilinear coordinates, the result is called a sinogram, similar in concept to displaying the Fourier spectrum. Like the Fourier spectrum, a sinogram contains the data necessary to reconstruct \( f(x, y) \).

![Figure 5.39](image)

**FIGURE 5.39** Two images and their sinograms (Radon transforms). Each row of a sinogram is a projection along the corresponding angle on the vertical axis. Image (c) is called the Shepp-Logan phantom. In its original form, the contrast of the phantom is quite low. It is shown enhanced here to facilitate viewing.

**Figure 5.39** (b) is the sinogram of the rectangle shown in **Figure 5.39** (a).

**Figure 5.39** (c) shows an image of the Shepp-Logan phantom, a widely used synthetic image designed to simulate the absorption of major areas of the brain. The sinogram of **Figure 5.39** (c) is shown in **Figure 5.39** (d).
To obtain a formal expression for a back-projected image from Radon transform, referring to Figure 5.37, we begin with a single point, \( g(\rho, \theta_k) \), of the complete projection, \( g(\rho, \theta) \), for a fixed value of rotation, \( \theta_k \).

Forming part of an image by back-projecting this single point is simply to copy the line \( L(\rho, \theta_k) \) onto the image, where the value of each point in that line is \( g(\rho, \theta_k) \). Repeating this process of all values of \( \rho_j \) in the projected signal results

\[
\tilde{f}_{\theta_k}(x, y) = g(\rho, \theta_k) = g(x \cos \theta_k + y \sin \theta_k, \theta_k).
\]

This equation holds for an arbitrary value of \( \theta_k \), therefore, we can write in general that the image formed from a single backprojection obtained at an angle \( \theta \) is given by

\[
f_{\theta}(x, y) = g(x \cos \theta + y \sin \theta, \theta) \tag{5.11-5}
\]

We form the final image by integrating over all the back-projected images

\[
f(x, y) = \int_0^\pi f_{\theta}(x, y) d\theta \tag{5.11-6}
\]

In the discrete case, the integral becomes a sum of all back-projected images:

\[
f(x, y) = \sum_{\theta=0}^\pi f_{\theta}(x, y) \tag{5.11-7}
\]

For example, if 0.5° increments are being used, the summation is from 0 to 179.5°.

A back-projected image formed in this manner is referred to as a laminogram, which is only an approximation to the image from which the projections were generated.
Example 5.18: Obtaining back-projected images from sinograms

Equation

\[ f(x, y) = \sum_{\theta=0}^{\pi} f_b(x, y) \]  \hspace{1cm} (5.11-7)

was used to generate the back-projected images in Figure 5.32 through Figure 5.34 from projections obtained with

\[ g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho). \]  \hspace{1cm} (5.11-4)

These equations were also used to generate Figure 5.40 (a) and Figure 5.40 (b), which show the back-projected images corresponding to the sinograms in Figure 5.39 (b) and Figure 5.39 (d).

Note that there is a significant amount of blurring shown in Figure 5.40 (a) and (b). It is obvious that a straight use of Equations (5.11-4) and (5.11-7) will not yield acceptable results.
The Fourier-Slice Theorem

The relationship relating the 1-D Fourier transform of a projection and the 2-D Fourier transform of the region from which the projection was obtained is the basis for reconstruction methods capable of dealing with the blurring problem.

The 1-D Fourier transform of a projection with respect to $\rho$ is

$$G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) e^{-j2\pi \rho \omega} d\rho$$  \hspace{1cm} (5.11-8)

where $\omega$ is the frequency variable, and this expression is for a given value of $\theta$.

Substituting

$$g(\rho, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho) dx dy$$  \hspace{1cm} (5.11-3)

for $g(\rho, \theta)$ results the expression

$$G(\omega, \theta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - \rho) e^{-j2\pi \rho \omega} dx dy d\rho$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \left[ \int_{-\infty}^{\infty} \delta(x \cos \theta + y \sin \theta - \rho) e^{-j2\pi \rho \omega} d\rho \right] dx dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \omega(x \cos \theta + y \sin \theta)} dx dy$$  \hspace{1cm} (5.11-9)

By letting $u = \omega \cos \theta$ and $v = \omega \sin \theta$, (5.11-9) becomes

$$G(\omega, \theta) = \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi (ux + vy)} dx dy \right]_{u=\omega \cos \theta; v=\omega \sin \theta}$$  \hspace{1cm} (5.11-10)

We recognize (5.11-10) as the 2-D Fourier transform of $f(x, y)$ evaluated at the values of $u$ and $v$ indicated.
Equation (5.11-10) leads to

\[ G(\omega, \theta) = [F(u, v)]_{u=\omega \cos \theta; v=\omega \sin \theta} = F(\omega \cos \theta, \omega \sin \theta), \quad (5.11-11) \]

which is known as the Fourier-slice theorem (or the projection-slice theorem).

The Fourier-slice theorem states that the Fourier transform of a projection is a slice of the 2-D Fourier transform of the region from which the projection was obtained.

This terminology can be explained with Figure 5.41.

As Figure 5.41 shows, the 1-D Fourier transform of an arbitrary projection is obtained by extracting the values of \( F(u, v) \) along a line oriented at the same angle as the angle used in generating the projection.

In principle, we could obtain \( f(x, y) \) simply by obtaining the inverse Fourier transform \( F(u, v) \), though it is expensive computationally with the involvement of inverting a 2-D transform.
Reconstruction Using Parallel-Beam Filtered Backprojections

Regarding to the blurred results, fortunately, there is a simple solution based on filtering the projections before computing the backprojections.

Recall

\[ f(t, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(\mu, \nu)e^{j2\pi(\mu t + \nu z)}d\mu d\nu, \quad (4.5-8) \]

the 2-D inverse Fourier transform of \( F(u, v) \) is

\[ f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux + vy)}dudv. \quad (5.11-12) \]

As in (5.11-10) and (5.11-11), letting \( u = \omega \cos \theta \) and \( v = \omega \sin \theta \), we can express (5.11-12) in polar coordinates:

\[ f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} F(\omega \cos \theta, \omega \sin \theta)e^{j2\pi(\omega x \cos \theta + y \sin \theta)}\omega d\omega d\theta \quad (5.11-13) \]

Then, using the Fourier-slice theorem, we have

\[ f(x, y) = \int_{0}^{2\pi} \int_{0}^{\infty} G(\omega, \theta)e^{j2\pi(\omega x \cos \theta + y \sin \theta)}\omega d\omega d\theta. \quad (5.11-14) \]

Using the fact that \( G(\omega, \theta + \pi) = G(-\omega, \theta) \), we can express (5.11-14) as

\[ f(x, y) = \int_{0}^{\pi} \int_{-\infty}^{\infty} |\omega| G(\omega, \theta)e^{j2\pi(\omega x \cos \theta + y \sin \theta)}d\omega d\theta. \quad (5.11-15) \]

In terms of integration with respect to \( |\omega| \), the term \( x \cos \theta + y \sin \theta \) is a constant, which is recognized as \( \rho \). Thus, (5.11-15) can be written as

\[ f(x, y) = \int_{0}^{\pi} \left[ \int_{-\infty}^{\infty} |\omega| G(\omega, \theta)e^{j2\pi\rho}d\omega \right]_{\rho = x \cos \theta + y \sin \theta} d\theta. \quad (5.11-16) \]
Recall
\[ f(t) = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi \mu t} d\mu, \] (4.2-17)

the inner expression in (5.11-16) is a 1-D inverse Fourier transform with the added term \(|\omega|\).

Based on the discussion in Section 4.7, \(|\omega|\) is a one-dimensional filter function.

\(|\omega|\) is not integrable, because its amplitude extends to \(+\infty\) in both directions, so the inverse Fourier transform is undefined.

In practice, the approach is to window the ramp so it becomes zero outside of defined frequency interval, as shown in Figure 5.42 (a).

Figure 5.42 (b) shows its spatial domain representation, obtained by computing its inverse Fourier transform. The resulting windowed filter exhibits noticeable ringing in the spatial domain. As discussed in Chapter 4, windowing with a smooth function will help in this situation.
An M-point discrete window function used frequently for implementation with the 1-D FFT is given by

\[
    h(\omega) = \begin{cases} 
        c + (c - 1) \cos \frac{2\pi \omega}{M - 1} & 0 \leq \omega \leq (M - 1) \\
        0 & \text{otherwise}
    \end{cases} \quad (5.11-17)
\]

When \( c = 0.54 \), this function is called the Hamming window.

Figure 5.42 (c) is a plot of the Hamming window, and Figure 5.42 (d) shows the product of this window and the band-limited ramp filter shown in Figure 5.42 (a).

Figure 5.42 (e) shows the representation of the product in the spatial domain, obtained by computing the inverse FFT.

Comparing Figure 5.42 (e) and Figure 5.42 (b), we can find that ringing was reduced in the window ramp.

On the other hand, because the width of the central lobe in Figure 5.42 (e) is slightly wider than that of Figure 5.42 (b), we would expect backprojections based on a Hamming window to have less ringing but be slightly more blurred.

Recalling

\[
    G(\omega, \theta) = \int_{-\infty}^{\infty} g(\rho, \theta) e^{-j2\pi \rho} d\rho \quad (5.11-8)
\]

that \( G(\omega, \theta) \) is the 1-D Fourier transform of \( g(\rho, \theta) \), which is a single projection obtained at a fixed angle, \( \theta \).
Equation
\[ f(x, y) = \int_0^\pi \left[ \int_{-\infty}^{\infty} \omega | G(\omega, \theta) e^{j2\pi \rho_d} d\omega \right] e^{j\rho \cos \theta + y \sin \theta} d\theta \]  

(5.11-16)

states that the complete, back-projected image \( f(x, y) \) is obtained as follows:

1. Compute the 1-D Fourier transform of each projection.
2. Multiply each Fourier transform by the filter function \(|\omega|\), which has been multiplied by a suitable (e.g., Hamming) window.
3. Obtain the inverse 1-D Fourier transform of each resulting filtered transform.
4. Integrate (sum) all the 1-D inverse transform from Step 3.

This image reconstruction approach is called filtered backprojection.

In practice, because the data are discrete, all frequency domain computations are carried out using a 1-D FFT algorithm, and filtering is implemented using the same basic procedure explained in Chapter 4 for 2-D functions.
Example 5.19: Image reconstruction using filtered backprojections

Figure 5.43 (a) shows the rectangle reconstructed using a ramp filter. The most vivid feature of this result is the absence of any visually detectable blurring. However, ringing is present, visible as faint lines, especially around the corners of the rectangle. Figure 5.43 (c) can show these lines in the zoomed section.

Using a Hamming window on the ramp filter helped considerably with the ringing problem, at the expense of slight blurring, as Figure 5.43 (b) and Figure 5.43 (d) show.
The reconstructed phantom images shown in Figure 5.44 are from using the un-windowed ramp filter and a Hamming window on the ramp filter.

Since the phantom image does not have transitions that are sharp and prominent as the rectangle, so ringing is imperceptible in this case, though result shown in Figure 5.44 (b) is a slightly smooth than that of Figure 5.44 (a).

The discussion has been based on obtaining filtered backprojections via an FFT implementation. However, from the convolution theorem introduced in Chapter 4, we know that the equivalent results can be obtained using spatial convolution.

Note that the term inside the brackets in

\[ f(x, y) = \int_0^\pi \left[ \int_{-\infty}^{\infty} |\omega| G(\omega, \theta) e^{j2\pi\omega \rho} \, d\omega \right] e^{j2\pi(x \cos \theta + y \sin \theta)} \, d\theta \]  

(5.11-16)

is the inverse Fourier transform of the product of two frequency domain functions. According to the convolution theorem, they are equal to the convolution of the spatial representations (inverse Fourier transform) of these two functions.

Let \( s(\rho) \) denote the inverse Fourier transform of \(|\omega|\), we can write (5.11-16) as
\[
f(x, y) = \int_0^\pi \left[ \int_{-\infty}^{\infty} |\omega| G(\omega, \theta) e^{j2\pi\omega^0} d\omega \right]_{\rho = x \cos \theta + y \sin \theta} d\theta
\]
\[
= \int_0^\pi \left[ s(\rho) \star g(\rho, \theta) \right]_{\rho = x \cos \theta + y \sin \theta} d\theta
\]
\[
= \int_0^\pi \left[ \int_{-\infty}^{\infty} g(\rho, \theta) s(x \cos \theta + y \sin \theta - \rho) d\rho \right] d\theta
\]

The last two lines of (5.11-18) say the same thing: Individual backprojections at an angle \(\theta\) can be obtained by convolving the corresponding projection, \(g(\rho, \theta)\), and the inverse Fourier transform of the ramp filter, \(s(\rho)\).

With the exception of round off differences in computation, the results of using convolution will be identical to the results using FFT.

In general, convolution turns out to be more computationally efficient and is used in most of modern CT systems, while Fourier transform plays a central role in theoretical formulations and algorithm development.